

EE421/521
Image Processing

Lecture 12a
IMAGE RECONSTRUCTION

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Introduction

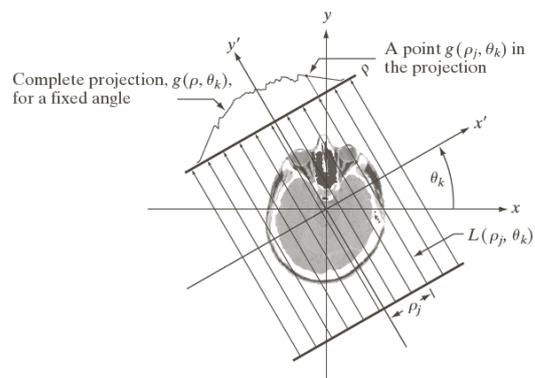
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Image Recovery

- Image Restoration
 - Undoing imaging degradations
- Image Reconstruction
 - Filling in for missing data

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Reconstruction from CT Projections



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Reconstruction from Fourier Phase

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Reconstruction from Magnitude-Only and Phase-Only Images

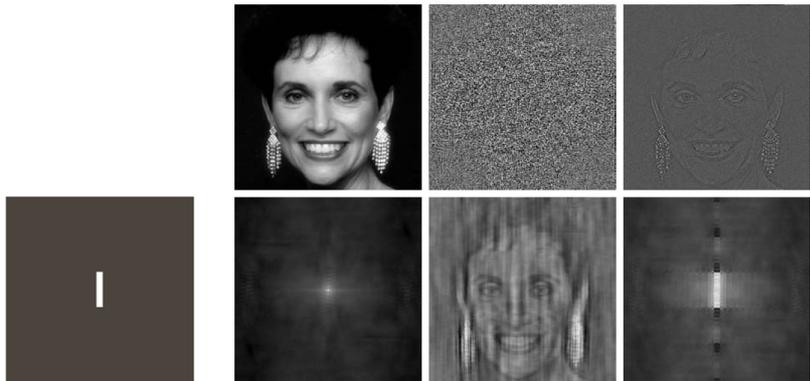


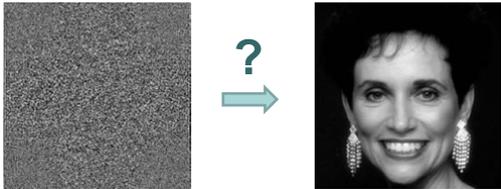
Fig.4.24(a), (f)

a b c
d e f

FIGURE 4.27 (a) Woman. (b) Phase angle. (c) Woman reconstructed using only the phase angle. (d) Woman reconstructed using only the spectrum. (e) Reconstruction using the phase angle corresponding to the woman and the spectrum corresponding to the rectangle in Fig.4.24(a). (f) Reconstruction using the phase of the rectangle and the spectrum of the woman.

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Reconstruction from Phase-Only Image



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Finite Image Support Assumption

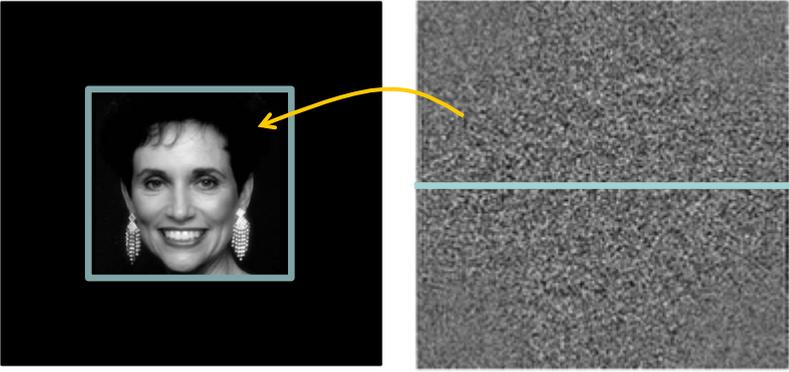


Image with finite support

Observed Fourier phase

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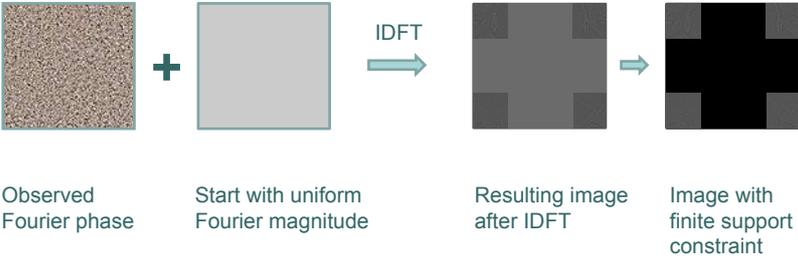
Why Reconstruction is Possible



Unknown intensities in less than **half** of the image

Unique information in **half** of the Fourier phase 9

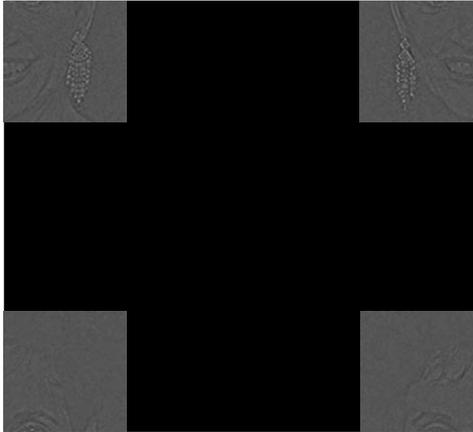
Reconstruction Algorithm: Initialization



Observed Fourier phase + Start with uniform Fourier magnitude $\xrightarrow{\text{IDFT}}$ Resulting image after IDFT \rightarrow Image with finite support constraint

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Finite Support Constraint



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Reconstruction Algorithm: Iterations

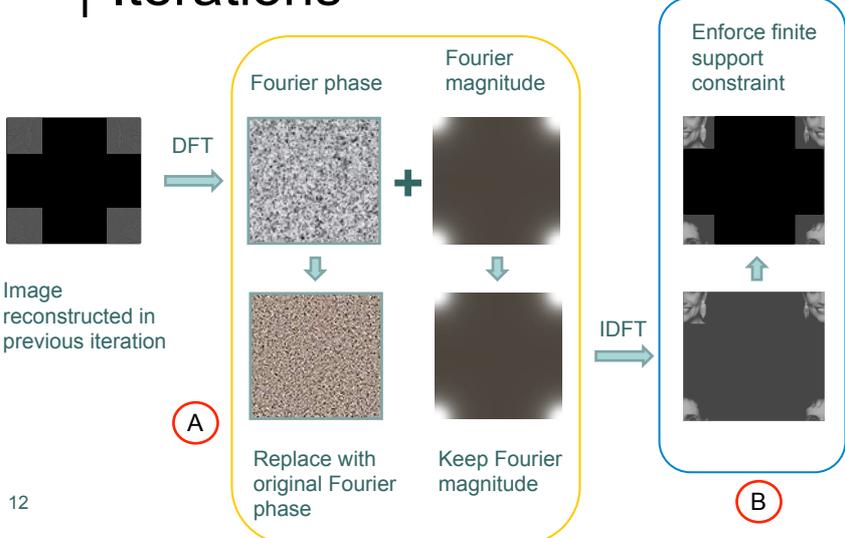


Image reconstructed in previous iteration

DFT

Fourier phase + Fourier magnitude

Replace with original Fourier phase

Keep Fourier magnitude

Enforce finite support constraint

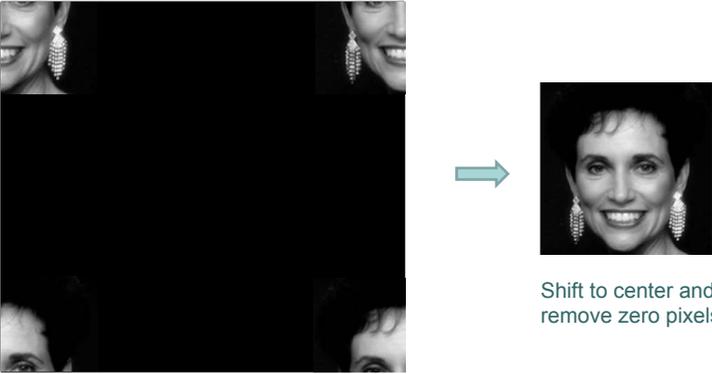
IDFT

A

B

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● ● ● | After Several Iterations of
A & B...



Shift to center and
remove zero pixels

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*Reconstruction
from CT
Projections*

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Computed Tomography

The diagram on the left shows a CT scanner setup. A 'Source' emits a fan beam that passes through a 'Subject' (represented as a circle with a cross) and is captured by a 'Detector' array. The diagram on the right illustrates the projection geometry. It shows a circular object in a coordinate system with axes x and y . A projection line is drawn at an angle θ_k from the x -axis. The distance from the origin to the projection line is $L(\rho_j, \theta_k)$. A point $g(\rho_j, \theta_k)$ is marked on the projection line. The complete projection for a fixed angle is shown as a series of parallel lines.

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Can We Recover the Image from its Projections?

The diagram illustrates the relationship between an absorption profile and its projections. The top part shows a square 'Absorption profile' with a central white circle. A 'Beam' of parallel rays passes through it. A 'Detector strip' on the right captures the resulting 'Ray' profiles, which are horizontal lines of varying intensity. The bottom part shows the same absorption profile with a vertical detector strip, resulting in vertical ray profiles. The final image on the right is a cross-shaped projection, representing the combination of horizontal and vertical projections.

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Reconstruction via Backprojections

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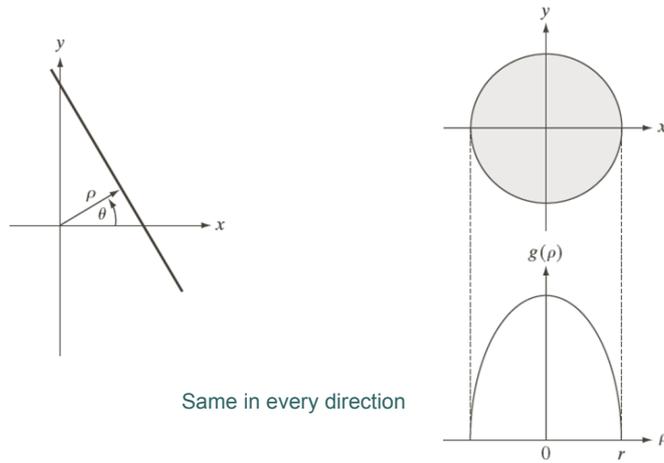
Blurring Caused by Backprojections

a b c
d e f

FIGURE 5.34 (a) A region with two objects. (b)–(d) Reconstruction using 1, 2, and 4 backprojections 45° apart. (e) Reconstruction with 32 backprojections 5.625° apart. (f) Reconstruction with 64 backprojections 2.8125° apart.

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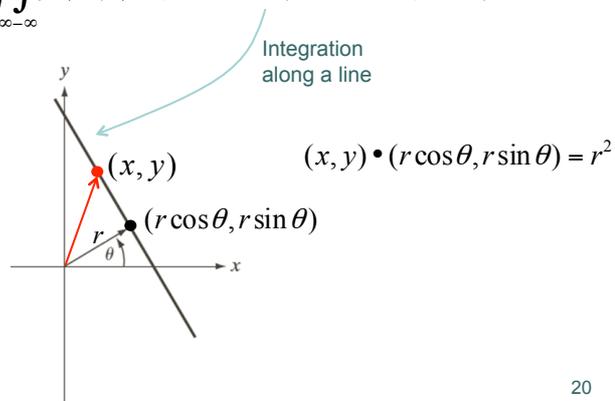
● ● ● | Projection of a Disk



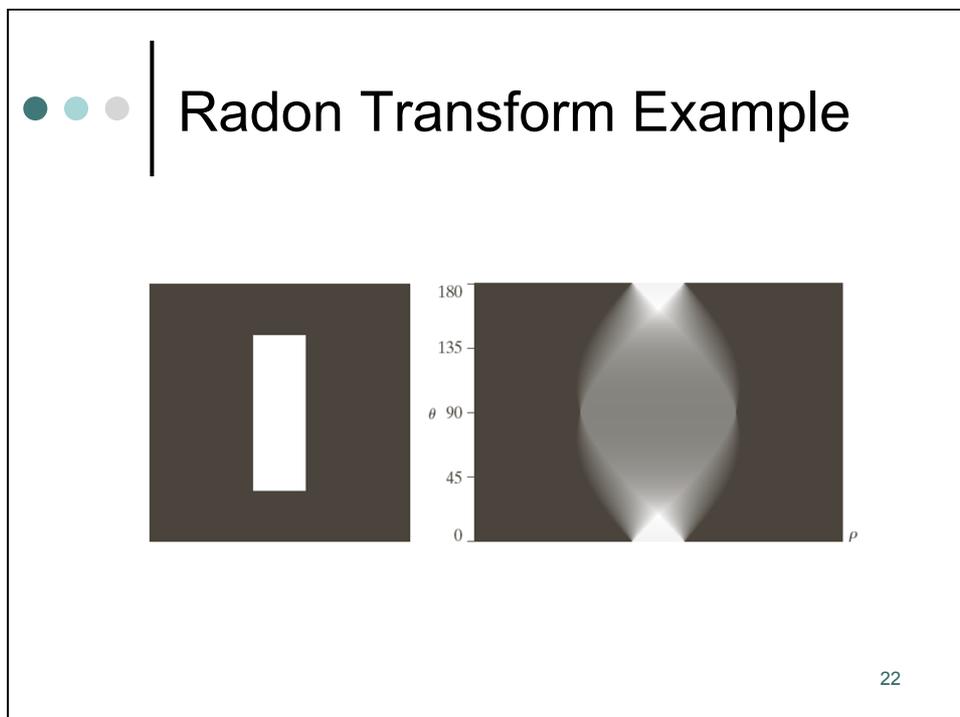
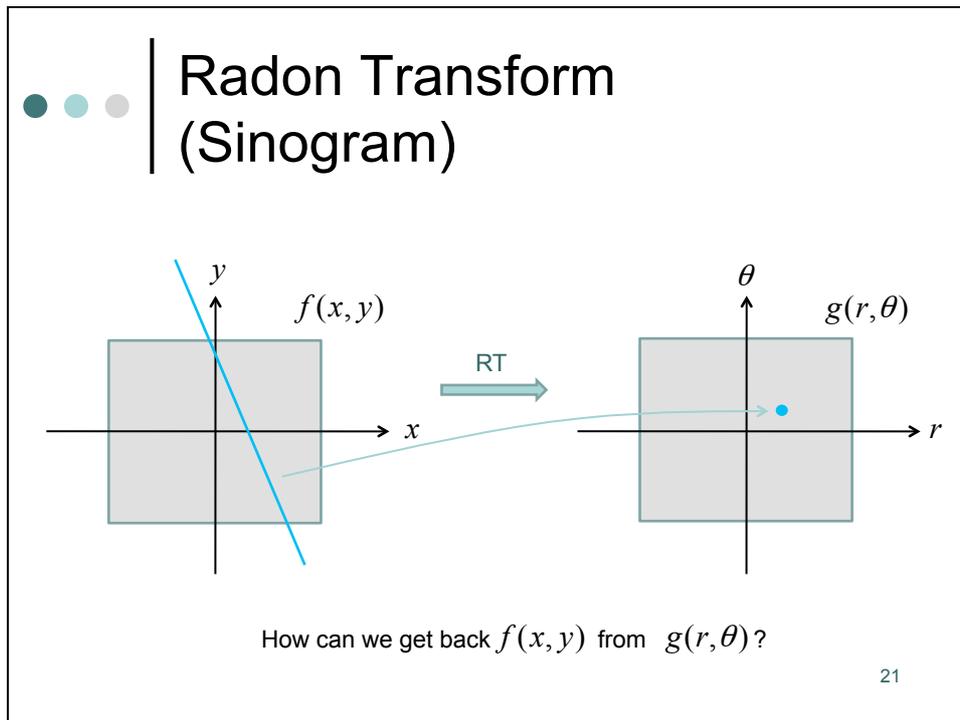
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● ● ● | Series of Projections: Radon Transform

$$g(r, \theta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \delta(x \cos \theta + y \sin \theta - r) dx dy$$

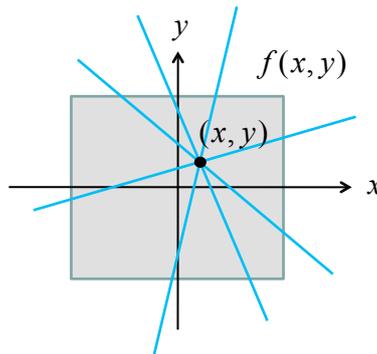


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Reconstruction with Backprojections

$$\hat{f}(x, y) = \int_{-\pi/2}^{\pi/2} \int_{-\infty}^{\infty} g(r, \theta) \delta(x \cos \theta + y \sin \theta - r) dr d\theta$$



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Reconstruction with Backprojections

$$\hat{f}(x, y) = \int_{-\pi/2}^{\pi/2} \int_{-\infty}^{\infty} g(r, \theta) \delta(x \cos \theta + y \sin \theta - r) dr d\theta$$

$$\hat{f}(x, y) = \int_{-\pi/2}^{\pi/2} g(x \cos \theta + y \sin \theta, \theta) d\theta$$

$$\hat{f}(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x', y') \int_{-\pi/2}^{\pi/2} \delta((x' - x) \cos \theta + (y' - y) \sin \theta) d\theta dx dy$$

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Reconstruction with Backprojections

$$\hat{f}(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x', y') \int_{-\pi/2}^{\pi/2} \delta((x' - x) \cos \theta + (y' - y) \sin \theta) d\theta dx' dy'$$

$$\hat{f}(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x', y') ((x - x')^2 + (y - y')^2)^{-1/2} dx' dy'$$

$$\hat{f}(x, y) = f(x, y) ** \frac{1}{\sqrt{x^2 + y^2}}$$

Causes blurring



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Approach 1: Fourier Slice Theorem

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Fourier Transform of Projections

- Compute the 1-D Fourier transform of the Radon transform with respect to r .

$$G(\omega, \theta) = \int_{-\infty}^{\infty} g(r, \theta) e^{-j\omega r} dr$$

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Fourier Transform of Projections

$$\begin{aligned}
 G(\omega, \theta) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \delta(x \cos \theta + y \sin \theta - r) e^{-j\omega r} dr dx dy \\
 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \int_{-\infty}^{\infty} \delta(x \cos \theta + y \sin \theta - r) e^{-j\omega r} dr dx dy \\
 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-j\omega x \cos \theta} e^{-j\omega y \sin \theta} dx dy \\
 &= F(\omega \cos \theta, \omega \sin \theta)
 \end{aligned}$$

A rotated line

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Fourier Slice (Projection Slice) Theorem

The 1-D Fourier transform of a projection is a slice of the 2-D Fourier transform

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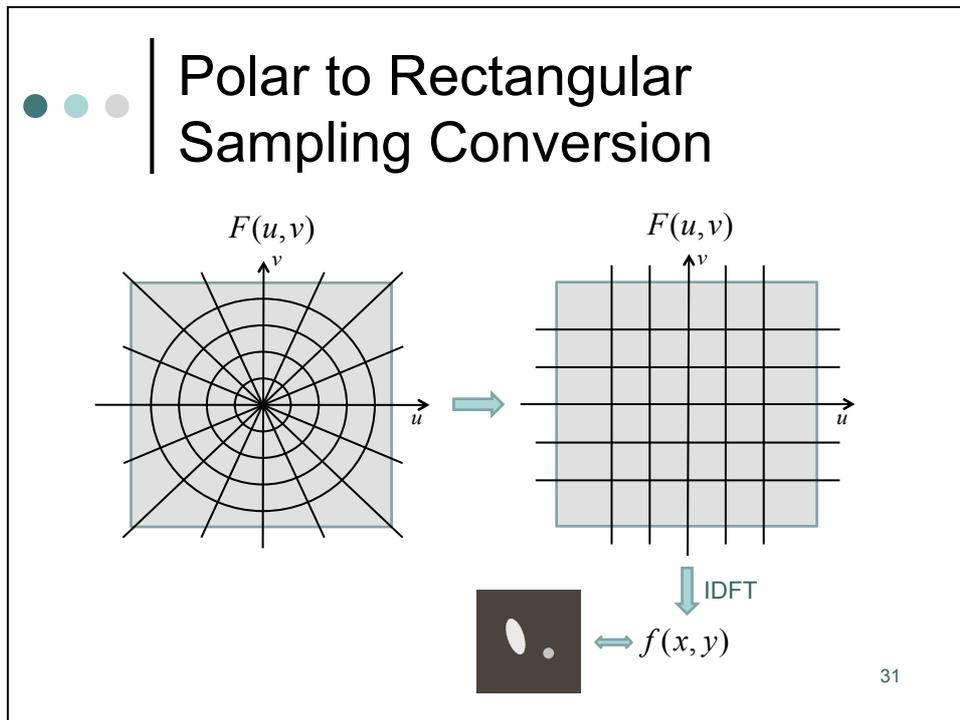
Polar Sampling

$F(u, v)$ at $u = \omega \cos \theta, v = \omega \sin \theta$

Uniform sampling along ω, θ

Polar sample locations for u, v

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***Approach 2:
Filtered
Backprojections***

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Reconstruction Equations

$$\begin{aligned}
 f(x, y) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u, v) e^{jux} e^{jvy} du dv \\
 &= \int_{-\pi}^{\pi} \int_{-\infty}^{\infty} F(\omega \cos \theta, \omega \sin \theta) e^{j\omega(x \cos \theta + y \sin \theta)} \omega d\omega d\theta \\
 &= \int_{-\pi}^{\pi} \int_{-\infty}^{\infty} \omega G(\omega, \theta) e^{j\omega(x \cos \theta + y \sin \theta)} d\omega d\theta
 \end{aligned}$$

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Correct Reconstruction from Backprojections

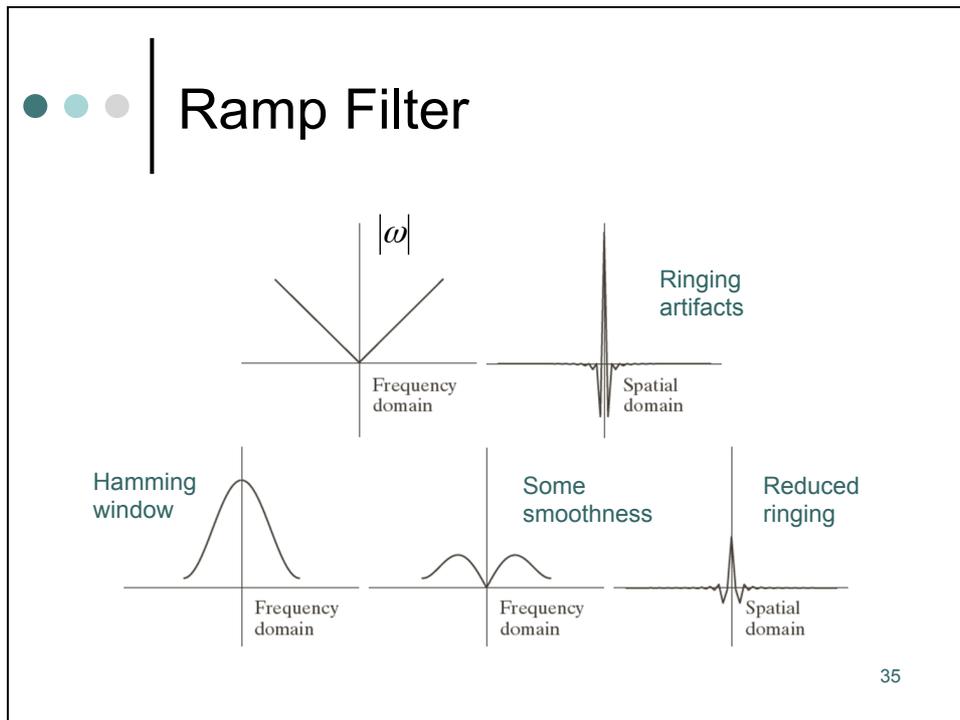
$$f(x, y) = \int_{-\pi/2}^{\pi/2} \left(\int_{-\infty}^{\infty} |\omega| G(\omega, \theta) e^{j\omega(x \cos \theta + y \sin \theta)} d\omega \right) d\theta$$

$$g(r, \theta) \Rightarrow G(\omega, \theta) \Rightarrow |\omega| G(\omega, \theta) \Rightarrow g'(r, \theta)$$



$$\hat{f}(x, y) = \int_{-\pi/2}^{\pi/2} g'(x \cos \theta + y \sin \theta, \theta) d\theta$$

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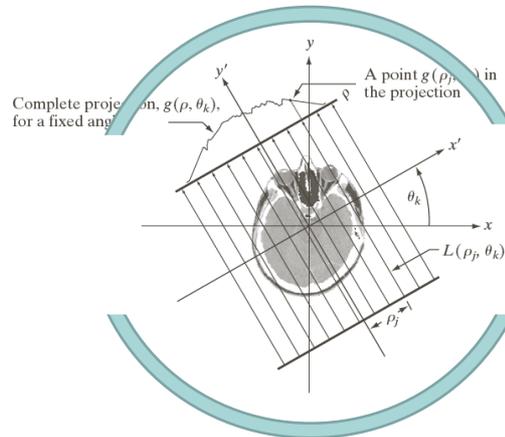
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Reconstruction from Partial Projections

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Missing Projections

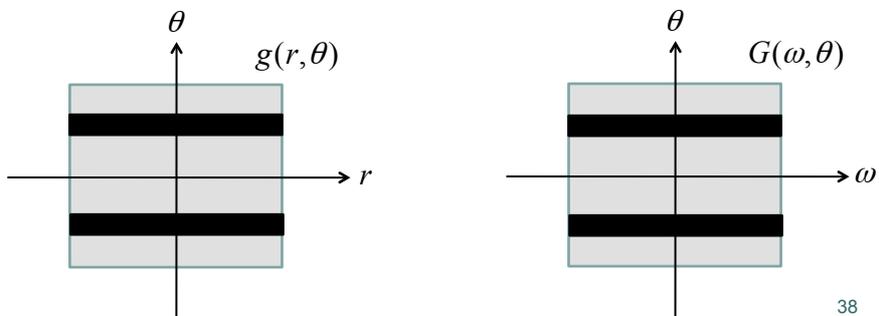


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Missing Angles in the Radon Transform

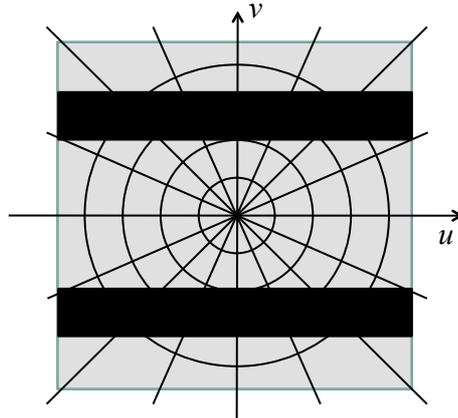
$$G(\omega, \theta) = \int_{-\infty}^{\infty} g(r, \theta) e^{-j\omega r} dr$$



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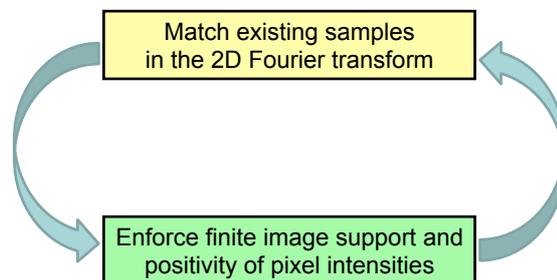
Missing Samples in the 2D Fourier Transform

$$F(u,v) \text{ at } u = \omega \cos \theta, v = \omega \sin \theta$$



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POCS Solution



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Project 3.3a

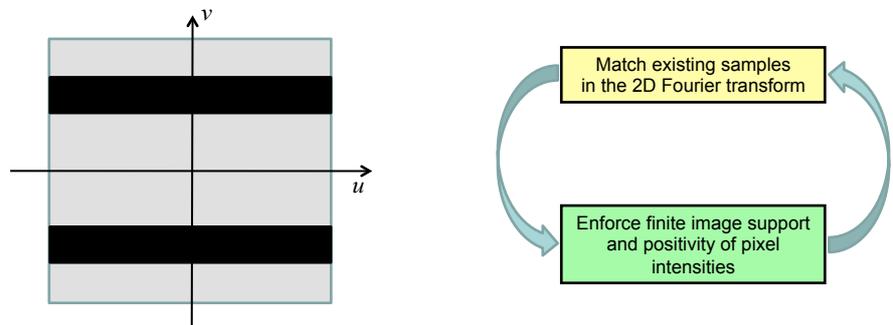
Image Reconstruction

Due 12.01.2013 Sunday

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Image Reconstruction via POCS



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Problem 3.3a

- Select a 256x256 monochrome image and display it.
- Pad zeros to the right and to the bottom of this image and then take its 512x512 DFT—call it $D(k,l)$.
- Define $D'(k,l)$ to be the observed DFT of the image with missing segments:

$$D'(k,l) = \begin{cases} 0 & 50 \leq k \leq 150 \\ 0 & 362 \leq k \leq 462 \\ D(k,l) & \text{elsewhere} \end{cases}$$

- Take the inverse DFT of $D'(k,l)$ and apply the “256x256 support” and “0–255 amplitude” constraint.
- Take the DFT of the image obtained in the previous step and match the observed samples of $D'(k,l)$.
- Repeat the previous two steps 5 times and display the results after each iteration. Comment on the results.

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Next Lecture

- IMAGE COMPRESSION

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